Direct approach to the study of soliton perturbations of the nonlinear Schrödinger equation and the sine-Gordon equation

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Based on the method of separation of variables, a direct approach in the study of soliton perturbations has been developed in our previous paper [Phys. Rev. E **54**, 6816 (1996)]. In this paper, we use it to deal with the nonlinear Schrödinger and the sine-Gordon equations under the action of perturbations. Results which differ from those in past papers are obtained. [S1063-651X(98)00807-1]

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I. INTRODUCTION

It is commonly known that there are several important exactly integrable nonlinear evolution equations which provide effective mathematical models for some very general physical phenomena. They are the Korteweg-de Vries (KdV), nonlinear Schrödinger (NLS), sine-Gordon (SG), and some other equations. As a matter of fact, in real physical applications, these equations usually come from some asymptotic expansion, so they are actually approximate equations. In more realistic situations, when higher-order terms must be taken into account, the equations we derived differ slightly from the standard ones by small additional terms that are called perturbations. Considerable attention was given to this aspect of soliton science, and various methods were developed by many authors in the past decades. A method based on the inverse scattering transformation (IST) [1–6] is very powerful when dealing with these cases. However, it is rather sophisticated and inconvenient for one who is not familiar with IST. Ostrovskii and his colleagues [7,8] first developed a direct approach to the study of soliton perturbations. Some general features of this approach can be seen from subsequent papers [9–19]. In this approach, perturbed nonlinear equations are usually linearized by expanding their solutions about the unperturbed ones. The most important technique is to find eigenfunctions of a linearized operator associated with the linearized equation. These eigenfunctions are used either to construct a Green's function [12], to invert the linearized equations [18,19], or in other alternative ways for the purpose of calculating the first-order corrections.

However, in Refs. [11,18], it is not difficult to find that the eigenfunctions are actually derived by making use of some knowledge of IST. In addition, the authors consider SG equations in characteristic coordinates. As to the study of SG equations in laboratory coordinates, authors generally adopt the so-called quasistationary assumption [10,15,17]. This leads to results which are valid only for a short time. And, in this scheme, the parameters variation on ''slow'' time scales is not taken into account. Therefore, to our knowledge, there is still not a satisfactory direct approach for SG soliton perturbations yet.

We have developed a direct approach for studying the perturbed KdV equation [20]. Since this scheme is a natural application of the classical perturbation theory and the general method of separation of variables, we believe that this scheme is easy to follow and use. In this paper, we use this approach to study the perturbed NLS and SG equations, and get results for first-order corrections for both equations.

II. SOLITON PERTURBATIONS OF NLS EQUATION

A. Linearization

Let us consider the perturbed NLS equation

$$iu_t+u_{xx}+2|u|^2u=i\epsilon R[u], \tag{1}$$

where the subscripts stand for partial differentiation with respect to the time t and the space x, ϵ is a small positive constant measuring the weakness of the perturbation ($0<\epsilon\ll 1$), and the perturbation term $R[u]$ is a known function of u, u_x, u_{xx},... . When $R=0$, Eq. (1) reduces into the standard NLS equation

$$iu_t+u_{xx}+2|u|^2u=0. \tag{2}$$

It is well known that Eq. (2) has the following single-soliton solution:

$$u(x,t)=2\beta\,\mathrm{sech}[2\beta(x-x_0+4\alpha t)]\times\exp[-2i\alpha x-4i(\alpha^2-\beta^2)t-i\theta_0], \tag{3}$$

where α, β, x_0, and θ_0 are four real parameters which determine the propagating velocity, height (as well as width), initial position, and initial phase of the soliton, respectively.

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Since what we study in this paper is the effects of perturbation on a single soliton, Eq. (1) is subject to the initial condition

$$u(x,0)=2\beta\ \mathrm{sech}[2\beta(x-x_0)]\exp(-2i\alpha x-i\theta_0). \quad (4)$$

At first, we linearize Eq. (1) following the lines of Ref. [18]. The independent variable t is transformed into several variables by

$$t_n=\epsilon^n t, \quad n=0,1,2,\ldots, \quad (5)$$

where each t_n is an order of ϵ smaller than the previous time. Thus the time derivatives should be replaced by the expansion

$$\partial_t=\partial_{t_0}+\epsilon\partial_{t_1}+\epsilon^2\partial_{t_2}+\cdots . \quad (6)$$

At the same time, u and $R[u]$ are expanded in an asymptotic series

$$u=u^{(0)}+\epsilon u^{(1)}+\epsilon^2 u^{(2)}+\cdots, \quad (7)$$

$$R[u]=R^{(1)}[u^0]+\epsilon R^{(2)}[u^{(0)},u^{(1)}]+\cdots . \quad (8)$$

Substituting Eqs. (6)–(8) into Eq. (1), and equating the coefficients of each power of ϵ, we obtain the following approximation equations of different orders:

$$iu_{t_0}^{(0)}+u_{xx}^{(0)}+2|u^{(0)}|^2u^{(0)}=0, \quad (9)$$

$$iu_{t_0}^{(1)}+u_{xx}^{(1)}+4|u^{(0)}|^2u^{(1)}+2[u^{(0)}]^2\overline{u}^{(1)}$$
$$=iR^{(1)}[u^{(0)}]-iu_{t_1}^{(0)},\ldots, \quad (10)$$

where the overbar signifies the complex conjugate. Meanwhile, the initial condition (4) should be replaced by

$$u^{(0)}(x,0)=2\beta\ \mathrm{sech}[2\beta(x-x_0)]\exp(-2i\alpha x-i\theta_0),$$
$$u^{(n)}(x,0)=0, \quad \text{for } n=1,2,\ldots . \quad (11)$$

The zeroth-order approximation equation (9) is just the standard NLS equation. It has a single-soliton solution that is formally the same as Eq. (3):

$$u^{(0)}(x,t_0)=2\beta e^{-i\theta}\mathrm{sech}\ z, \quad (12)$$

with

$$z=2\beta(x-\xi), \quad \xi_{t_0}=-4\alpha, \quad (13)$$

$$\theta=\alpha z/\beta+\delta=2\alpha(x-\xi)+\delta, \quad \delta_{t_0}=-4(\alpha^2+\beta^2). \quad (14)$$

Due to perturbation, the soliton parameters α, β, ξ, and δ are now supposed to be functions of the slow time variables $t_1,t_2,\ldots$, but α and β are independent of t_0, and the t_0 dependence of ξ and δ are given by the second equations in Eqs. (13) and (14), respectively. It follows from Eq. (12) that

$$u_{t_n}^{(0)}=e^{-i\theta}[(4i\alpha\beta\xi_{t_n}-2i\beta\delta_{t_n})\phi_1(z)-2i\alpha_{t_n}\phi_2(z)$$
$$+2\beta_{t_n}\psi_1(z)+4\beta^2\xi_{t_n}\psi_2(z)], \quad (15)$$

where

$$\phi_1(z)=\mathrm{sech}\ z, \quad \phi_2(z)=z\ \mathrm{sech}\ z, \quad (16)$$

$$\psi_1(z)=(1-z\ \tanh z)\mathrm{sech}\ z, \quad \psi_2(z)=\tanh z\,\mathrm{sech}\ z. \quad (17)$$

For the study of the perturbations on a single soliton, it is more convenient to use z (space variable in a coordinate system moving with the soliton) as a new independent variable in place of x. Then the linearized NLS equation (10), together with the appropriate initial conditions (11) are reduced into the following form with the aid of Eq. (15):

$$iu_{t_0}^{(1)}+8i\alpha\beta u_z^{(1)}+4\beta^2u_{zz}^{(1)}+4|u^{(0)}|^2u^{(1)}+2[u^{(0)}]^2\overline{u}^{(1)}$$
$$=iF^{(1)}\equiv iR^{(1)}[u^{(0)}]-ie^{-i\theta}[(4i\alpha\beta\xi_{t_1}-2i\beta\delta_{t_1})\phi_1(z)$$
$$-2i\alpha_{t_1}\phi_2(z)+2\beta_{t_1}\psi_1(z)+4\beta^2\xi_{t_1}\psi_2(z)], \quad (18)$$
$$u^{(1)}(z,0)=0.$$

Introducing the transformation

$$u^{(1)}=e^{-i\theta}v^{(1)}=e^{-i\theta}[A^{(1)}+iB^{(1)}], \quad (19)$$

where $A^{(1)}$ and $B^{(1)}$ are the real and imaginary part of $v^{(1)}$, respectively, one can rewrite the complex equation (18) into the following real simultaneous equations:

$$\begin{cases} A_{t_0}^{(1)}+4\beta^2\hat{L}_1B^{(1)}=\mathrm{Re}[R^{(1)}e^{i\theta}]-2\beta_{t_1}\psi_1(z)-4\beta^2\xi_{t_1}\psi_2(z), \\ B_{t_0}^{(1)}-4\beta^2\hat{L}_2A^{(1)}=\mathrm{Im}[R^{(1)}e^{i\theta}]-(4\alpha\beta\xi_{t_1}-2\beta\delta_{t_1})\phi_1(z)+2\alpha_{t_1}\phi_2(z), \end{cases} \quad (20)$$

with

$$A^{(1)}(z,0)=B^{(1)}(z,0)=0, \quad (21)$$

where

$$\hat{L}_1=\frac{d^2}{dz^2}+2\,\mathrm{sech}^2\ z-1, \quad \hat{L}_2=\frac{d^2}{dz^2}+6\,\mathrm{sech}^2\ z-1 \quad (22)$$

are two self-adjoint linear differential operators.

B. Eigenvalue problem

To find the solutions of Eq. (20) with initial condition (21) by the separation of variables, we can follow the same methods we used in Ref. [20]. But here we need to solve the following coupled eigenvalue problem:

$$\hat{L}_1\phi=\lambda\psi,\qquad \hat{L}_2\psi=\lambda\phi. \tag{23}$$

Obviously, the simultaneous equations (23) can easily be rewritten into the following two standard eigenvalue problem equations:

$$\hat{L}_2\hat{L}_1\phi=\lambda^2\phi, \tag{24}$$

$$\hat{L}_1\hat{L}_2\psi=\lambda^2\psi. \tag{25}$$

In Appendix A, the eigenfunction of Eq. (23) [of Eqs. (24) and (25), as well] for continuous eigenvalue $\lambda=-(k^2+1)$, $-\infty<k<\infty$ is derived in a way somewhat different than that in Ref. [20],

$$\phi(z,k)=\frac{1}{\sqrt{2\pi}(k^2+1)}(1-k^2-2ik\tanh z)e^{ikz}, \tag{26}$$

$$\psi(z,k)=\frac{1}{\sqrt{2\pi}(k^2+1)}(-1-k^2-2ik\tanh z+2\tanh^2 z)e^{ikz}=\frac{1}{ik}\phi_z(z,k), \tag{27}$$

and it is easy to check directly that $\hat{L}_2\hat{L}_1$ and $\hat{L}_1\hat{L}_2$ also have two eigenfunctions $\phi_1(z),\phi_2(z)$ and $\psi_1(z),\psi_2(z)$ for discrete eigenvalue $\lambda=0$, respectively. It must be pointed out emphatically that although $\phi_2(z)$ and $\psi_1(z)$ are solutions of Eqs. (24) and (25) for $\lambda=0$, respectively, they do not satisfy the simultaneous equations (23). In fact, it is easy to derive that

$$\hat{L}_1\phi_2(z)=-2\psi_2(z),\quad \hat{L}_2\psi_1(z)=2\phi_1(z), \tag{28}$$

which are different from Eq. (23) for $\lambda=0$. Obviously, the above eigenfunctions have the following symmetries which can be checked directly:

$$\overline{\phi}(z,k)=\phi(z,-k),\quad \overline{\psi}(z,k)=\psi(z,-k),$$
$$\overline{\phi}_j(z)=\phi_j(z),\quad \overline{\psi}_j(z)=\psi_j(z),\quad j=1,2. \tag{29}$$

Now we have introduced two sets of eigenfunctions $\{\phi\}=\{\phi(z,k),\ \phi_j(z);\ j=1,2\}$ and $\{\psi\}=\{\psi(z,k),\ \psi_j(z);\ j=1,2\}$. They constitute two complete sets of orthonormal bases. The orthonormalities and completeness of them are defined as follows

(i) Orthonormalities:

$$\int_{-\infty}^{\infty}\phi(z,k)\overline{\psi}(z,k')dz=\int_{-\infty}^{\infty}\psi(z,k)\overline{\phi}(z,k')dz=\delta(k-k'), \tag{30}$$

$$\int_{-\infty}^{\infty}\phi(z,k)\psi_j(z)dz=\int_{-\infty}^{\infty}\psi(z,k)\phi_j(z)dz=0,\quad j=1,2, \tag{31}$$

$$\int_{-\infty}^{\infty}\phi_j(z)\psi_l(z)dz=\delta_{jl},\quad j,l=1,2. \tag{32}$$

(ii) Completeness:

$$\int_{-\infty}^{\infty}\phi(z,k)\overline{\psi}(z',k)dk+\sum_{j=1}^{2}\phi_j(z)\psi_j(z')=\delta(z-z'). \tag{33}$$

Equations (30) and (33) are proved through a straightforward calculation with the aid of the residue theorem in Appendixes B and C, respectively. Equation (31) can also be proved in a similar way, while Eq. (32) can be checked directly.

C. Effects of perturbation on a soliton

In order to solve the initial-value problem (20) and (21) by the separation of variables, we expand $A^{(1)}(z,t_0)$ on the basis $\{\psi\}$ while $B^{(1)}(z,t_0)$ on $\{\phi\}$ as

$$A^{(1)}(z,t_0)=\int_{-\infty}^{\infty}a^{(1)}(t_0,k)\psi(z,k)dk+\sum_{j=1}^{2}a_j^{(1)}(t_0)\psi_j(z), \tag{34}$$

$$B^{(1)}(z,t_0)=\int_{-\infty}^{\infty}b^{(1)}(t_0,k)\phi(z,k)dk+\sum_{j=1}^{2}b_j^{(1)}(t_0)\phi_j(z). \tag{35}$$

Owing to Eq. (29), the coefficientsin in Eqs. (34) and (35) must satisfy the following relations to guarantee both $A^{(1)}$ and $B^{(1)}$ are real:

$$\overline{a}^{(1)}(t_0,k)=a^{(1)}(t_0,-k),\quad \overline{a}_j^{(1)}(t_0)=a_j^{(1)}(t_0),$$
$$\overline{b}^{(1)}(t_0,k)=b^{(1)}(t_0,-k),\quad \overline{b}_j^{(1)}(t_0)=b_j^{(1)}(t_0). \tag{36}$$

Substituting Eqs. (34) and (35) into Eqs. (20) and (21), and employing Eqs. (23) and (28), one obtains

$$\int_{-\infty}^{\infty}[\dot{a}^{(1)}(t_0,k)+4\beta^2\lambda b^{(1)}(t_0,k)]\psi(z,k)dk+\dot{a}_1^{(1)}(t_0)\psi_1(z)+[\dot{a}_2^{(1)}(t_0)-8\beta^2b_2^{(1)}(t_0)]\psi_2(z)=\mathrm{Re}[R^{(1)}e^{i\theta}]-2\beta_{t_1}\psi_1(z)-4\beta^2\xi_{t_1}\psi_2(z), \tag{37}$$

$$\int_{-\infty}^{\infty}[\dot{b}^{(1)}(t_0,k)-4\beta^2\lambda a^{(1)}(t_0,k)]\phi(z,k)dk+[\dot{b}_1^{(1)}(t_0)-8\beta^2a_1^{(1)}(t_0)]\phi_1(z)+\dot{b}_2^{(1)}(t_0)\phi_2(z)=\mathrm{Im}[R^{(1)}e^{i\theta}]-(4\alpha\beta\xi_{t_1}-2\beta\delta_{t_1})\phi_1(z)+2\alpha_{t_1}\phi_2(z), \tag{38}$$

with

$$a^{(1)}(0,k)=a_j^{(1)}(0)=b^{(1)}(0,k)=b_j^{(1)}(0)=0,\quad j=1,2, \tag{39}$$

where the overdot signifies derivative with respect to t_0. Making use of the orthonormal relations (30)–(32), we obtain from Eqs. (37)–(39) the following ordinary differential equations with zero-initial conditions:

$$\dot{a}^{(1)}(t_0,k)+4\beta^2\lambda b^{(1)}(t_0,k)=p^{(1)}(k),\quad a^{(1)}(0,k)=0, \tag{40}$$

$$\dot{b}^{(1)}(t_0,k)-4\beta^2\lambda a^{(1)}(t_0,k)=q^{(1)}(k),\quad b^{(1)}(0,k)=0, \tag{41}$$

$$\dot{a}_1^{(1)}(t_0)=p_1^{(1)}-2\beta_{t_1},\quad a_1^{(1)}(0)=0, \tag{42}$$

$$\dot{b}_2^{(1)}(t_0)=q_2^{(1)}+2\alpha_{t_1},\quad b_2^{(1)}(0)=0, \tag{43}$$

$$\dot{a}_2^{(1)}(t_0)-8\beta^2 b_2^{(1)}(t_0)=p_2^{(1)}-4\beta^2\xi_{t_1},\quad a_2^{(1)}(0)=0, \tag{44}$$

$$\dot{b}_1^{(1)}(t_0)-8\beta^2 a_1^{(1)}(t_0)=q_1^{(1)}-(4\alpha\beta\xi_{t_1}-2\beta\delta_{t_1}),\quad b_1^{(1)}(0)=0, \tag{45}$$

where

$$p^{(1)}(k)=\int_{-\infty}^{\infty}\mathrm{Re}[R^{(1)}e^{i\theta}]\overline{\phi}(z,k)dz,$$

$$p_j^{(1)}=\int_{-\infty}^{\infty}\mathrm{Re}[R^{(1)}e^{i\theta}]\overline{\phi_j}(z)dz,\quad j=1,2, \tag{46}$$

$$q^{(1)}(k)=\int_{-\infty}^{\infty}\mathrm{Im}[R^{(1)}e^{i\theta}]\overline{\psi}(z,k)dz,$$

$$q_j^{(1)}=\int_{-\infty}^{\infty}\mathrm{Im}[R^{(1)}e^{i\theta}]\overline{\psi_j}(z)dz,\quad j=1,2. \tag{47}$$

If $R^{(1)}e^{i\theta}$ is independent of θ (for example, if $R^{(1)}=-u^{(0)}$, $-u_{xx}^{(0)}$, etc.) so the right-hand side of Eqs. (40)–(45) are independent of t_0 in the moving coordinate system, Eqs. (42) and (43), will lead to secularity. In fact, integrating them over t_0 yields $a_1^{(1)}(t_0)=(p_1^{(1)}-2\beta_{t_1})t_0$ and $b_2^{(1)}(t_0)=(q_2^{(1)}+2\alpha_{t_1})t_0$, which grows infinitely in time. Thus we must demand that

$$p_1^{(1)}-2\beta_{t_1}=0\rightarrow a_1^{(1)}(t_0)=0, \tag{48}$$

$$q_2^{(1)}+2\alpha_{t_1}=0\rightarrow b_2^{(1)}(t_0)=0. \tag{49}$$

Due to Eqs. (48) and (49), Eqs. (44) and (45) also lead to secularity. Similarly, we demand that

$$p_2^{(1)}-4\beta^2\xi_{t_1}=0\rightarrow a_2^{(1)}(t_0)=0, \tag{50}$$

$$q_1^{(1)}-(4\alpha\beta\xi_{t_1}-2\beta\delta_{t_1})=0\rightarrow b_1^{(1)}(t_0)=0. \tag{51}$$

Now we begin to find out the effects of perturbation on the soliton, i.e., the t_1 dependence of soliton parameters and the first-order correction. At first, inserting Eqs. (48), (50) and (49), (51) into the second equations in Eqs. (46) and (47), respectively, we obtain the following four important formulas immediately:

$$\beta_{t_1}=\frac{1}{2}p_1^{(1)}=\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{Re}[R^{(1)}e^{i\theta}]\overline{\phi_1}(z)dz=\frac{1}{2}\mathrm{Re}\int_{-\infty}^{\infty}R^{(1)}e^{i\theta}\,\mathrm{sech}\,z\,dz, \tag{52}$$

$$\xi_{t_1}=\frac{1}{4\beta^2}p_2^{(1)}=\frac{1}{4\beta^2}\int_{-\infty}^{\infty}\mathrm{Re}[R^{(1)}e^{i\theta}]\overline{\phi_2}(z)dz=\frac{1}{4\beta^2}\mathrm{Re}\int_{-\infty}^{\infty}R^{(1)}e^{i\theta}z\,\mathrm{sech}\,z\,dz, \tag{53}$$

$$\alpha_{t_1}=-\frac{1}{2}q_2^{(1)}=-\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{Im}[R^{(1)}e^{i\theta}]\overline{\psi_2}(z)dz=-\frac{1}{2}\mathrm{Im}\int_{-\infty}^{\infty}R^{(1)}e^{i\theta}\tanh z\,\mathrm{sech}\,z\,dz, \tag{54}$$

$$\delta_{t_1}=2\alpha\xi_{t_1}-\frac{1}{2\beta}q_1^{(1)}=2\alpha\xi_{t_1}-\frac{1}{2\beta}\int_{-\infty}^{\infty}\mathrm{Im}[R^{(1)}e^{i\theta}]\overline{\psi_1}(z)dz=2\alpha\xi_{t_1}-\frac{1}{2\beta}\mathrm{Im}\int_{-\infty}^{\infty}R^{(1)}e^{i\theta}(1-z\tanh z)\mathrm{sech}\,z\,dz. \tag{55}$$

Equations (52)–(55) determine how the soliton shape and position are affected by the perturbation. They are the same as those obtained by inverse scattering perturbation theory [5] and other direct methods [18].

Secondly, we return to the simultaneous equations (40) and (41). Their solution can be easily derived in a standard way:

$$a^{(1)}(t_0,k)=-\frac{q^{(1)}(k)}{4\beta^2\lambda}[1-\cos(4\beta^2\lambda t_0)]+\frac{p^{(1)}(k)}{4\beta^2\lambda}\sin(4\beta^2\lambda t_0), \tag{56}$$

$$b^{(1)}(t_0,k)=\frac{p^{(1)}(k)}{4\beta^2\lambda}[1-\cos(4\beta^2\lambda t_0)]+\frac{q^{(1)}(k)}{4\beta^2\lambda}\sin(4\beta^2\lambda t_0). \tag{57}$$

Noting that $a_j^{(1)}(t_0)=b_j^{(1)}(t_0)=0$, $j=1,2$ [see Eqs. (48)–(51)], we get from Eqs. (34) and (35) that

$$A^{(1)}(z,t_0)=\int_{-\infty}^{\infty}a^{(1)}(t_0,k)\psi(z,k)dk, \tag{58}$$

$$B^{(1)}(z,t_0)=\int_{-\infty}^{\infty} b^{(1)}(t_0,k)\phi(z,k)dk. \tag{59}$$

Inserting Eqs. (58) and (59) into Eq. (19), we obtain

$$\begin{aligned}u^{(1)}(z,t_0)&=e^{-i\theta}[A^{(1)}(z,t_0)+iB^{(1)}(z,t_0)]\\&=e^{-i\theta}\int_{-\infty}^{\infty}dk[a^{(1)}(t_0,k)\psi(z,k)\\&\quad+ib^{(1)}(t_0,k)\phi(z,k)].\end{aligned} \tag{60}$$

Noting that the eigenfunctions for continuous spectrum can be rewritten as

$$\begin{aligned}\phi(z,k)&=\frac{-1}{\sqrt{2\pi}(k^2+1)}\,e^{ikz}[(k+i\ \tanh z)^2-\text{sech}^2\ z],\\ \psi(z,k)&=\frac{-1}{\sqrt{2\pi}(k^2+1)}\,e^{ikz}[(k+i\ \tanh z)^2+\text{sech}^2\ z],\end{aligned} \tag{61}$$

we finally get the following explicit expression for the first-order correction after some calculation:

$$\begin{aligned}u^{(1)}(z,t_0)=&\frac{-ie^{-i\theta}}{8\pi\beta^2}\int_{-\infty}^{\infty}dk\,\frac{\overline{I}(k)}{(k^2+1)^3}\,[1-e^{-4i\beta^2(k^2+1)t_0}]\\&\times(k+i\tanh z)^2e^{ikz}+\frac{ie^{-i\theta}}{8\pi\beta^2}\int_{-\infty}^{\infty}dk\,\frac{I(-k)}{(k^2+1)^3}\\&\times[1-e^{4i\beta^2(k^2+1)t_0}]\text{sech}^2\ z\ e^{ikz},\end{aligned} \tag{62}$$

where

$$I(k)=\int_{-\infty}^{\infty}dz[\overline{R}^{(1)}e^{-i\theta}(k+i\tanh z)^2-R^{(1)}e^{i\theta}\text{sech}^2\ z]e^{ikz}. \tag{63}$$

Keener and McLaughlin [11,12] and afterwards Herman [18] studied the soliton perturbations for the NLS equation. Herman declared that their results agree. Especially, Herman derived an exact explicit expression of the first-order correction for the NLS equation [given by Eq. (144) in Ref. [18]]. Obviously, our correction [given by Eq. (62)] is some different from that in Ref. [18]. To be exact, dividing Eq. (62) by β^2 and multiplying the integrand of the first term in Eq. (62) by k^2, we get Eq. (144) in Ref. [18]. A straightforward calculation shows that our $u^{(1)}(z,t_0)$ satisfies Eq. (10) and initial condition (11).

D. Damping NLS equation

As an important example, let us consider the so-called damping NLS equation in which $R[u]=-u$

$$iu_t+u_{xx}+2|u|^2u=-i\epsilon u. \tag{64}$$

Obviously,

$$R^{(1)}[u^{(0)}]=-u^{(0)}=-2\beta e^{-i\theta}\text{sech}\ z. \tag{65}$$

The time dependence of the soliton parameters can be easily obtained from Eqs. (52)–(55):

$$\beta_{t_1}=-\beta\int_{-\infty}^{\infty}\text{sech}^2\ z\,dz=-2\beta, \tag{66}$$

$$\xi_{t_1}=-\frac{1}{2\beta}\int_{-\infty}^{\infty}z\,\text{sech}^2\ z\,dz=0, \tag{67}$$

$$\alpha_{t_1}=\beta\int_{-\infty}^{\infty}\tanh z\,\text{sech}^2\ z\,dz=0, \tag{68}$$

$$\delta_{t_1}=0+\int_{-\infty}^{\infty}(1-z\tanh z)\text{sech}^2\ z\,dz=1. \tag{69}$$

Returning to the original time variable t (noting that $\partial_t=\partial_{t_0}+\epsilon\partial_{t_1}$ up to the first-order approximation) and performing the integrals, we rewrite Eqs. (66)–(69) as

$$\beta_t=-2\epsilon\beta,\ \ \xi_t=-4\alpha,\ \ \delta_t=-4(\alpha^2+\beta^2),\ \ \alpha_t=0, \tag{70}$$

which lead to

$$\begin{aligned}&\beta=\beta_0e^{-2\epsilon t},\ \ \alpha=\alpha_0,\ \ \xi=\xi_0-4\alpha_0t,\\&\delta=\delta_0-(4\alpha_0^2-\epsilon)t+\beta_0^2e^{-4\epsilon t}/\epsilon,\end{aligned} \tag{71}$$

where β_0, α_0, ξ_0, and δ_0 are all constants. Equation (71) means that the height of soliton dampens (while the width increases) with time exponentially, while the propagating velocity is not affected by the perturbation.

To derive the first-order correction $u^{(1)}(z,t_0)$, we first get from Eqs. (46) and (47) that

$$p^{(1)}(k)=\int_{-\infty}^{\infty}\text{Re}[R^{(1)}e^{i\theta}]\overline{\phi}(z,k)dz=\sqrt{2\pi}\beta\,\text{sech}(\pi k/2), \tag{72}$$

$$q^{(1)}(k)=\int_{-\infty}^{\infty}\text{Im}[R^{(1)}e^{i\theta}]\overline{\psi}(z,k)dz=0, \tag{73}$$

where the integrals were performed with the aid of a residue theorem similar to the one in Appendix B. Then substituting Eqs. (72) and (73) into Eqs. (56) and (57) yields

$$a^{(1)}(t_0,k)=\sqrt{2\pi}\,\text{sech}(\pi k/2)\sin(4\beta^2\lambda t_0)/4\beta\lambda, \tag{74}$$

$$b^{(1)}(t_0,k)=\sqrt{2\pi}\,\text{sech}(\pi k/2)[1-\cos(4\beta^2\lambda t_0)]/4\beta\lambda. \tag{75}$$

Note that the soliton parameter β should be taken as a constant β_0 up to the first-order approximation. Substituting Eqs. (74) and (75) into Eq. (60), and replacing t_0 by t, we finally obtain the first-order correction

$$u^{(1)}(z,t)=\sqrt{2\pi}e^{-i\theta}\int_{-\infty}^{\infty}dk\,\mathrm{sech}(\pi k/2)\sin(4\beta_0^2\lambda t)\times\psi(z,k)/4\beta_0\lambda+i\sqrt{2\pi}e^{-i\theta}\int_{-\infty}^{\infty}dk\;\mathrm{sech}(\pi k/2)\times[1-\cos(4\beta_0^2\lambda t)]\phi(z,k)/4\beta_0\lambda. \quad (76)$$

III. SOLITON PERTURBATIONS OF SG EQUATION

A. Linearization

Now we return to the perturbed SG equation

$$u_{tt}-u_{xx}+\sin u=\epsilon R[u]. \quad (77)$$

It is well known that the standard SG equation $(R=0)$ has a single-soliton solution

$$u_0(x,t)=4\arctan\exp[m(x-at)],\quad m=1/\sqrt{1-a^2}. \quad (78)$$

Equation (77) can be linearized in the same way as was done in Sec. II:

$$u^{(0)}_{t_0t_0}-u^{(0)}_{xx}+\sin u^{(0)}=0, \quad (79)$$

$$u^{(1)}_{t_0t_0}-u^{(1)}_{xx}+\cos u^{(0)}u^{(1)}=R[u^{(0)}]-2u^{(0)}_{t_0t_1},\ldots \quad (80)$$

and the initial condition for perturbations on a single soliton is

$$u^{(0)}(x,0)=4\arctan e^{mx}, \quad (81)$$

$$u^{(n)}(x,0)=0,\quad \text{for } n=1,2,\ldots . \quad (82)$$

Obviously, Eq. (79) has a single soliton solution formally the same as Eq. (78)

$$u^{(0)}(x,t_0)=4\arctan e^z,\quad z=m(x-\xi),\quad \xi_{t_0}=a, \quad (83)$$

where the soliton parameters m and ξ are supposed to be functions of the slow time variables $t_1,t_2,\ldots$, by virtue of perturbation. Then the second term on the right-hand side of Eq. (80) is replaced by

$$u^{(0)}_{t_0t_1}=-2am_{t_1}(1-z\tanh z)\mathrm{sech}\, z-2m^2a\xi_{t_1}\tanh z\;\mathrm{sech}\, z. \quad (84)$$

Noting that Eq. (83) gives rise to $\cos u^{(0)}=1-2\,\mathrm{sech}^2 z$, we rewrite Eq. (80) as

$$u^{(1)}_{t_0t_0}-u^{(1)}_{xx}+(1-2\,\mathrm{sech}^2 z)u^{(1)}=F^{(1)}(z)\equiv R[u^{(0)}]+4am_{t_1}(1-z\tanh z)\mathrm{sech}\, z+4m^2a\xi_{t_1}\tanh z\,\mathrm{sech}\, z. \quad (85)$$

In a coordinate system moving with the soliton, Eq. (85) and initial conditions (82) become

$$u^{(1)}_{t_0t_0}-2mau^{(1)}_{t_0z}-u^{(1)}_{zz}+(1-2\,\mathrm{sech}^2 z)u^{(1)}=F^{(1)}, \quad (86)$$

$$u^{(1)}(z,0)=0,\quad u^{(1)}_{t_0}(z,0)=0, \quad (87)$$

respectively. To solve the initial-value problem (86) and (87), we must make a further transformation on the independent variable to eliminate the first or second term of Eq. (86). For instance, in this paper, we may introduce a new time variable

$$\tau=t_0/2m-(1+a)z/2. \quad (88)$$

Equation (86) is replaced by

$$u^{(1)}_{\tau z}-u^{(1)}_{zz}+(1-2\,\mathrm{sech}^2 z)u^{(1)}=F^{(1)}. \quad (89)$$

According to the separation of variables, we must search for special solutions of Eq. (89), which can be expressed as production of a function of τ with a function of z

$$u^{(1)}(z,\tau)=T(\tau)X(z), \quad (90)$$

and at the same time, we suppose that

$$F^{(1)}(z,\tau)=f(\tau)X_z(z). \quad (91)$$

Substituting Eqs. (90) and (91) into Eq. (86), we reduce Eq. (86) into two ordinary differential equations

$$X_{zz}+(2\,\mathrm{sech}^2 z-1)X=\lambda' X_z, \quad (92)$$

$$T_\tau-\lambda' T=f, \quad (93)$$

where λ' is a constant (eigenvalue) to be determined.

B. Eigenvalue problem

Equation (92) is just a generalized eigenvalue problem equation

$$\hat{L}X=\lambda' X_z,\quad \hat{L}=\partial_{zz}+2\,\mathrm{sech}^2 z-1, \quad (94)$$

which differs from the ordinary cases with X_z on the right-hand side of Eq. (94) taking the place of X. $\hat{L}$ is nothing but the self-adjoint operator $\hat{L}_1$ introduced in Sec. II. Recalling that $\psi(z,k)=\phi_z(z,k)/ik$ [see Eq. (27)], one can easily find from the first equation in Eq. (23) that

$$\hat{L}\phi(z,k)=\lambda'\phi_z(z,k)\quad\text{for}\quad \lambda'=i(k+k^{-1}),\quad -\infty<k<\infty, \quad (95)$$

$$\hat{L}\phi_1(z)=0, \quad (96)$$

which means that $\{\phi\}$ can also be used as a basis for the SG equation.

C. Effects of perturbation on a soliton

Following the lines of the above section, we expand $u^{(1)}(z,\tau)$ on the basis $\{\phi\}$:

$$u^{(1)}(z,\tau)=\int_{-\infty}^{\infty}T(\tau,k)\phi(z,k)dk+\sum_{j=1}^{2}T_j(\tau)\phi_j(z). \quad (97)$$

Substituting Eq. (97) into Eq. (89), and employing Eqs. (95), (96), and (28), one can obtain

$$\int_{-\infty}^{\infty} ik[\dot{T}(\tau,k)-\lambda' T(\tau,k)]\psi(z,k)dk+\dot{T}_2(\tau)\psi_1(z)$$
$$-[\dot{T}_1(\tau)-2T_2(\tau)]\psi_2(z)=F^{(1)}, \tag{98}$$

which gives rise to the following ordinary differential equations with the aid of orthonormal relations (30)–(32):

$$\dot{T}(\tau,k)-\lambda' T(\tau,k)=\frac{1}{ik}\int_{-\infty}^{\infty} F^{(1)}(z)\overline{\phi}(z,k)dz, \tag{99}$$

$$\dot{T}_2(\tau)=\int_{-\infty}^{\infty} F^{(1)}(z)\phi_1(z)dz', \tag{100}$$

$$\dot{T}_1(\tau)-2T_2(\tau)=-\int_{-\infty}^{\infty} F^{(1)}(z)\phi_2(z)dz. \tag{101}$$

If $R[u(z)]$ does not contain time explicitly, the right-hand side of Eqs. (99)–(101) should be independent of τ. Then Eqs. (100) and (101) will gives rise to the secularities, and the nonsecular condition is

$$\int_{-\infty}^{\infty} F^{(1)}(z)\phi_1(z)dz=0\rightarrow\dot{T}_2(\tau)=0, \tag{102}$$

$$\int_{-\infty}^{\infty} F^{(1)}(z)\phi_2(z)dz=0\rightarrow\dot{T}_1(\tau)=0 \quad \text{and} \quad T_2(\tau)=0. \tag{103}$$

So we get from Eqs. (102) and (103) that $T_2(\tau)=0$, $T_1(\tau)=\text{const}$. As was pointed out above, the nonsecular conditions (102) and (103) will determine the slow time dependence of the soliton parameters

$$m_{t_1}=-\frac{1}{4a}\int_{-\infty}^{\infty} R[u^{(0)}]\text{sech } z\,dz, \tag{104}$$

$$\xi_{t_1}=-\frac{1}{4m^2a}\int_{-\infty}^{\infty} R[u^{(0)}]z\text{ sech } z\,dz. \tag{105}$$

Equations (104) and (105) are consistent with those obtained by the inverse scattering perturbation theory. To our knowledge, they have not been obtained by any other direct approach yet. Now we begin to derive the first-order correction $u^{(1)}$. At first, we solve Eq. (99) in a standard way

$$T(\tau,k)=c(k)e^{\lambda'\tau}-\frac{1}{ik\lambda'}\int_{-\infty}^{\infty} F^{(1)}(z)\overline{\phi}(z,k)dz, \tag{106}$$

where the coefficient $c(k)$ should be determined by the initial conditions. Inserting Eqs. (106) and (88) into Eq. (97), and noting that $T_1(\tau)=T_1$ is a constant, we obtain the general solution of Eq. (86) as follows:

$$u^{(1)}(z,t_0)=-\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}dz'\,\frac{1}{ik\lambda'}\,\phi(z,k)\overline{\phi}(z',k)F^{(1)}(z')$$
$$+\int_{-\infty}^{\infty}dk\,c(k)\exp\{\lambda'[t_0-m(1+a)z]/2m\}$$
$$\times\phi(z,k)+T_1\phi_1(z). \tag{107}$$

The coefficients $c(k)$ and T_1 in Eq. (107) can be determined by the initial conditions (87) through a series of calculations:

$$c(k)=\frac{1}{ik\lambda'}\int_{-\infty}^{\infty}dz\,\overline{\phi}(z,k)F^{(1)}(z)\exp[m\lambda'(1+a)z], \tag{108}$$

$$T_1=-\frac{1}{4m^2}\int_{-\infty}^{\infty}dz\,z\,\phi_2(z)F^{(1)}(z). \tag{109}$$

Substituting Eqs. (108) and (109) into Eq. (107), one finally obtains the first-order correction as follows:

$$u^{(1)}(z,t_0)=\int_{-\infty}^{\infty}\frac{dk}{ik\lambda'}\,\phi(z,k)\int_{-\infty}^{\infty}dz'\,\overline{\phi}(z',k)F^{(1)}(z')$$
$$\times\left[\exp\left\{\frac{\lambda'}{2m}[t_0-m(1+a)(z-z')]\right\}-1\right]$$
$$-\frac{1}{4m^2}\,\phi_1(z)\int_{-\infty}^{\infty}dz'z'\,\phi_2(z')F^{(1)}(z'). \tag{110}$$

It is not difficult to check through a straightforward calculation that $u^{(1)}(z,t_0)$ really satisfies Eq. (86) and initial conditions (87). In contrast to KdV and SG equations, here $u^{(1)}(z,t_0)$ also contains the contribution made by the discrete spectrum eigenfunctions. Obviously, Eq. (110) can be further rewritten as

$$u^{(1)}(z,t_0)=\int_{-\infty}^{\infty}G(z,t_0;z')F^{(1)}(z')dz', \tag{111}$$

where $G(z,t_0;z')$ is the Green's function defined by

$$G(z,t_0;z')=\int_{-\infty}^{\infty}\frac{dk}{ik\lambda'}\,\phi(z,k)\overline{\phi}(z',k)$$
$$\times\left\{\exp\left[\frac{\lambda'}{2m}[t_0-m(1+a)(z-z')]\right]-1\right\}$$
$$-\frac{1}{4m^2}\,\phi_1(z)z'\,\phi_2(z'). \tag{112}$$

IV. CONCLUSION

In this paper, we studied the soliton perturbations for NLS and SG equations by a direct approach, and obtained some results for the above two equations. At first, the first-order correction for the NLS equation is somewhat different than that obtained by other authors [18]. A careful calculation shows that our correction really satisfies the linearized equation while that in Ref. [18] does not. So we suspect that there are probably some errors in the calculation in Ref. [18].

In the past papers mentioned above, the authors dealt with a time-dependent operator associated with the linearized equation directly. An elaborate scheme is developed to derive the eigenfunctions of this operator by making use of some knowledge of IST. In our scheme, since the operator is time independent, the derivation of eigenfunctions is more direct and easier. In addition, the separation of time and space actually bring some convenience for the subsequent derivation. For instance, the secular terms appears clearly. An unexpected thing appears in this study: the expansion basis for the above two different equations is the same. We believe that this must suggest some deep connection between them.

In summary, this scheme is totally independent of IST, and derivations differ from the past papers substantially. It is helpful to have it as an alternative way to study soliton perturbations.

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APPENDIX A: DERIVATION OF EIGENFUNCTIONS $\phi(Z,K)$ AND $\psi(Z,K)$

Let us consider the eigenvalue problem

$$\hat{L}_1\phi=\lambda\psi,\quad \hat{L}_2\psi=\lambda\phi, \tag{A1}$$

where $\hat{L}_1=d^2/dz^2+2\,\text{sech}^2 z-1$ and $\hat{L}_2=d^2/dz^2+6\,\text{sech}^2 z-1$. Along the lines of Ref. [20], we assume that

$$\phi(z,k)=e^{ikz}\rho(z,k),\quad \psi(z,k)=e^{ikz}\sigma(z,k), \tag{A2}$$

where $\rho(z,k)$ and $\sigma(z,k)$ are supposed to have the asymptotic behavior $\rho(z,k)\to\text{const}$ and $\sigma(z,k)\to\text{const}$ as $z\to\pm\infty$. Then the asymptotic equation of Eq. (A1) leads to $\lambda=-(k^2+1)$. Inserting Eq. (A2) into Eq. (A1), one obtains an ordinary differential equation system for ρ and σ

$$\hat{L}_1\rho+2ik\rho_z-k^2\rho+(k^2+1)\sigma=0,\quad \hat{L}_2\sigma+2ik\sigma_z-k^2\sigma+(k^2+1)\rho=0. \tag{A3}$$

To determine $\rho(z,k)$ and $\sigma(z,k)$, we may expand each of them into a power series of ik as was done in Ref. [20]. However, we would like to use a method a bit different from that used in Ref. [20] here, in which ρ and σ are expanded into the following series:

$$\rho(z,k)=a_0+a_1\tanh z+\frac{a_2}{\cosh^2 z}+a_3\frac{\sinh z}{\cosh^3 z}+\frac{a_4}{\cosh^4 z}+a_5\frac{\sinh z}{\cosh^5 z}+\cdots, \tag{A4}$$

$$\sigma(z,k)=b_0+b_1\tanh z+\frac{b_2}{\cosh^2 z}+b_3\frac{\sinh z}{\cosh^3 z}+\frac{b_4}{\cosh^4 z}+b_5\frac{\sinh z}{\cosh^5 z}+\cdots, \tag{A5}$$

where the coefficients a_j and b_j, $j=1,2,...$, are functions of k to be determined. Inserting Eqs. (A4) and (A5) into Eq. (A3), and comparing the coefficient of 1, $\tanh z$, $1/\cosh^2 z$, $\sinh z/\cosh^3 z$,..., one obtains a series of algebraic equation systems which determine all the coefficients successively:

$$a_0=b_0, \tag{A6}$$

$$a_1=b_1, \tag{A7}$$

$$2a_0+2ika_1+(3-k^2)a_2-4ika_3+(k^2+1)b_2=0,\quad 6b_0+2ikb_1+(3-k^2)b_2-4ikb_3+(k^2+1)a_2=0, \tag{A8}$$

$$-4ika_2+(3-k^2)a_3+(k^2+1)b_3=0,\quad 4b_1-4ikb_2+(3-k^2)b_3+(k^2+1)a_3=0, \tag{A9}$$

$$-4a_2+6ika_3+(15-k^2)a_4-8ika_5+(k^2+1)b_4=0,\quad 6ikb_3+(15-k^2)b_4-8ikb_5+(k^2+1)a_4=0. \tag{A10}$$

If we assume that $a_i=b_j=0$, for $j\geqslant 3$, then it is easy to derive from Eqs. (A6)–(A10) that $a_2=0$, and the nonzero coefficients are

$$a_0=b_0=(1-k^2)c,\quad a_1=b_1=-2ikc,\quad b_2=-2c, \tag{A11}$$

where c is a constant determined by the orthonormality (30) or completeness (33) as

$$c=1/\sqrt{2\pi}(k^2+1). \tag{A12}$$

Inserting Eqs. (A4) and (A5) into Eq. (A2) and employing Eq. (A11) gives rise to

$$\phi(z,k)=\frac{1}{\sqrt{2\pi}(k^2+1)}\,e^{ikz}(1-k^2-2ik\tanh z), \tag{A13}$$

$$\psi(z,k)=\frac{1}{\sqrt{2\pi}(k^2+1)}\,e^{ikz}(1-k^2-2ik\tanh z-2/\cosh^2 z)=\frac{1}{\sqrt{2\pi}(k^2+1)}\,e^{2ik}(-1-k^2-2ik\tanh z+2\tanh^2 z). \tag{A14}$$

APPENDIX B: ORTHONORMALITY OF SETS $\{\phi\}$ AND $\{\psi\}$

Before the proof of the orthonormality relations, we must derive some useful integral formulas. At first let us calculate the following integral by the aid of the residue theorem:

$$I_1(k)=\int_{-\infty}^{\infty}e^{ikz}\tanh z\,dz. \tag{B1}$$

To do this, we consider a complex integral along a closed path c in plane $\zeta=z+i\eta$ as follows:

$$\oint_c f_1(\zeta)d\zeta=\oint_c e^{ik\zeta}\tanh\zeta\,d\zeta. \tag{B2}$$

Since the factor tanh ζ in the integrand is a periodic function with an imaginary period $i\pi$, we choose c to be the boundary of a rectangular region infinitely long: $-\infty<z<\infty$, $0\leqslant\eta\leqslant\pi$. In this region the integrand is analytic except for a simple pole $\zeta_0=i\pi/2$. We note that $e^{ik\zeta}$ oscillates rapidly for large z. Then the integral along the two straight line segments $z=\pm\infty$, $0\leqslant\eta\leqslant\pi$ should be zero. It follows that

$$\oint_c f_1(\zeta)d\zeta=[1-e^{-k\pi}]I_1(k). \tag{B3}$$

On the other hand, according to the residue theorem, we have

$$\oint_c f_1(\zeta)d\zeta=2\pi i\ \mathrm{Res}\ f_1(\zeta_0). \tag{B4}$$

The residue is easily obtained by the standard method

$$\mathrm{Res}\ f_1(\zeta_0)=\lim_{\zeta\to\zeta_0}(\zeta-\zeta_0)f_1(\zeta)=e^{-k\pi/2}. \tag{B5}$$

Comparing Eqs. (B3), (B4), and (B5), we get immediately

$$I_1(k)=\int_{-\infty}^{\infty}e^{ikz}\tanh z\,dz=i\pi/\sinh(\pi k/2). \tag{B6}$$

Starting from Eq. (B6), and employing the technique of integration by parts repeatedly, we obtain the following formulas successively:

$$I_2(k)=\int_{-\infty}^{\infty}e^{ikz}\frac{1}{\cosh^2 z}\,dz=\pi k/\sinh(\pi k/2), \tag{B7}$$

$$I_3(k)=\int_{-\infty}^{\infty}e^{ikz}\frac{\sinh z}{\cosh^3 z}\,dz=i\pi k^2/2\sinh(\pi k/2). \tag{B8}$$

In addition, some more integral formulas can also be obtained in a similar way,

$$I_4(k)=\int_{-\infty}^{\infty}e^{ikz}\frac{1}{\cosh z}\,dz=\pi/\cosh(\pi k/2), \tag{B9}$$

$$I_5(k)=\int_{-\infty}^{\infty}e^{ikz}\frac{\sinh z}{\cosh^2 z}\,dz=ik\pi/\cosh(\pi k/2), \tag{B10}$$

$$I_6(k)=\int_{-\infty}^{\infty}e^{ikz}\frac{1}{\cosh^3 z}\,dz=(1+k^2)\pi/2\cosh(\pi k/2), \tag{B11}$$

$$I_7(k)=\int_{-\infty}^{\infty}e^{ikz}\frac{z}{\cosh z}\,dz=i\pi^2/2\cosh(\pi k/2)-i\pi^2e^{-\pi k/2}/2\cosh^2(\pi k/2), \tag{B12}$$

$$I_8(k)=\int_{-\infty}^{\infty}e^{ikz}\frac{z\sinh z}{\cosh^2 z}\,dz=(1-\pi k/2)\pi/\cosh(\pi k/2)+\pi^2ke^{-\pi k/2}/2\cosh^2(\pi k/2), \tag{B13}$$

$$I_9(k)=\int_{-\infty}^{\infty}e^{ikz}\frac{z}{\cosh^3 z}\,dz=-i\pi k/\cosh(\pi k/2)+i\pi^2(1+k^2)\sinh(\pi k/2)/4\cosh^2(\pi k/2). \tag{B14}$$

Now we return to the orthogonality relations. As an example, let us calculate the following integral in detail:

$$\begin{aligned}\int_{-\infty}^{\infty}\phi(z,k)\overline{\psi}(z,k')dz&=\frac{1}{2\pi(k^2+1)[(k')^2+1]}\int_{-\infty}^{\infty}e^{i(k-k')z}[1-k^2-2ik\tanh z]\\&\quad\times[1-(k')^2+2ik'\tanh z-2\,\mathrm{sesh}^2 z]dz\\&=\frac{1}{2\pi(k^2+1)[(k')^2+1]}\int_{-\infty}^{\infty}e^{i(k-k')z}[(k^2-1)[(k')^2-1]+4kk']dz\\&\quad+\frac{1}{2\pi(k^2+1)[(k')^2+1]}[2i[k(k')^2-k^2k'-k+k']I_1(k-k')\\&\quad+[2(k^2-1)-4kk']I_2(k-k')+4ikI_3(k-k')].\end{aligned} \tag{B15}$$

Obviously, the first term of the second equation in Eq. (B15) gives rise to a δ function $\delta(k-k')$, while the other terms are just eliminated, namely,

$$2i[k(k')^2-k^2k'-k+k']I_1(k-k') +[2(k^2-1)-4kk']I_2(k-k')+4ikI_3(k-k')=0, \tag{B16}$$

which can be easily checked through a straightforward calculation with the aid of Eqs. (B6)–(B8). Thus the proof of Eq. (30) has been finished. Moreover, Eq. (31) can also be proved in the same way with the aid of Eqs. (B9)–(B14).

APPENDIX C: COMPLETENESS OF SETS $\{\phi\}$ AND $\{\psi\}$

We start from the integral

$$\begin{aligned}\int_{-\infty}^{\infty}\phi(z,k)\overline{\psi}(z',k)dk &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{(k^2+1)^2}\,e^{ik(z-z')}[1-k^2-2ik\tanh z]\\ &\quad\times[1-k^2+2ik\ \tanh z'-2\ \text{sesh}^2z']\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}dke^{ik(z-z')}-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{k^2+1}\,e^{ik(z-z')}\\ &\quad\times[4-2ik(\tanh z-\tanh z')-2\ \text{sesh}^2z']\\ &\quad+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{(k^2+1)^2}\,e^{ik(z-z')}4[1-ik\tanh z]\\ &\quad\times[ik\ \tanh z'+\tanh z'].\end{aligned} \tag{C1}$$

Obviously, the first term of the second equation in Eq. (C1) gives $\delta(z-z')$, while the other two terms can be calculated by the aid of the residue theorem. Note that the integrands in the above two terms as functions of complex variable k are analytical everywhere except for two poles $\pm i$ (of first and second order, respectively). The residues can be easily calculated in the standard way. Then the sums of the residues of the second and third terms are obtained as

$$\begin{aligned}\text{Res}(\pm i)=\pm i[&\text{sech}\ z(1-z'\ \tanh z')\text{sech}\ z'\\ &+z\ \text{sech}\ z\ \tanh z'\ \text{sech}\ z',\end{aligned} \tag{C2}$$

through a series of calculations. Finally, according to Jordan's lemma, we have

$$\begin{aligned}\int_{-\infty}^{\infty}\phi(z,k)\overline{\psi}(z',k)dk &= \delta(z-z')\pm 2\pi i\ \text{Res}(\pm i)\\ &= \delta(z-z')-\sum_{j=1}^{2}\phi_j(z)\psi_j(z'),\end{aligned} \tag{C3}$$

which is the same as the completeness relation (34).

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